

ASYMPTOTIC FORMULA FOR A PARTITION FUNCTION OF REVERSIBLE COAGULATION-FRAGMENTATION PROCESSES

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ABSTRACT

We construct a probability model seemingly unrelated to the considered stochastic process of coagulation and fragmentation. By proving for this model the local limit theorem, we establish the asymptotic formula for the partition function of the equilibrium measure for a wide class of parameter functions of the process. This formula proves the conjecture stated in [5] for the above class of processes. The method used goes back to A. Khintchine.

1. Introduction and summary

The motivation for our research came from a conjecture stated in [5], p. 462, in the following setting.

For a given integer N , denote by

$$(1.1) \quad \eta = (n_1, \dots, n_N) : 0 \leq n_k \leq N, \quad \sum_{k=1}^N kn_k = N$$

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a partition of N into n_k groups of size k , $k = 1, 2, \dots, N$ and by $\Omega_N = \{\eta\}$ the set of all partitions of N .

We will be interested in the particular probability measure μ on Ω_N given by

$$(1.2) \quad \mu_N(\eta) = C_N \frac{a_1^{n_1} a_2^{n_2} \dots a_N^{n_N}}{n_1! n_2! \dots n_N!}, \quad \eta = (n_1, \dots, n_N) \in \Omega_N,$$

where $a_k > 0$, $k = 1, 2, \dots, N$ and $C_N = C_N(a_1, \dots, a_N)$ is the partition function of the distribution μ_N :

$$(1.3) \quad C_N := C_N^{-1} = \sum_{\eta \in \Omega_N} \frac{a_1^{n_1} a_2^{n_2} \dots a_N^{n_N}}{n_1! n_2! \dots n_N!}, \quad \eta = (n_1, \dots, n_N) \in \Omega_N, \quad N = 1, 2, \dots$$

The probability measure μ_N is the equilibrium state of a class of reversible coagulation–fragmentation processes (CFP's) (see [5] for references).

CFP's trace their history from Smoluchowski (1916) and they have been intensively studied since this date. The process models the stochastic evolution in time of a population of N particles distributed into groups that coagulate and fragment at different rates. The model arises in different contexts of application: polymer kinetics, astrophysics, aerosols, biological phenomena such as animal grouping, blood cell aggregation, etc. Observe that particular choices of a_k , $k = 1, 2, \dots, N$ in (1.2) lead to a variety of known stochastic models. For example, when $a_k = \beta/k$, $k = 1, 2, \dots, N$, $\beta > 0$, (1.2) becomes the widely known Ewens sampling formula that arises in population genetics.

Following [5], we view CFP as a continuous-time Markov process on the state space Ω_N . Formally, a CFP is given by the rates ψ and ϕ of the two possible transitions: coagulation and fragmentation respectively. Namely, $\psi(i, j)$, $2 \leq i + j \leq N$ is the rate of merging of two groups of sizes i and j into one group of size $i + j$, and $\phi(i, j)$, $2 \leq i + j \leq N$ is the rate of splitting of a group of size $i + j$ into two groups of sizes i and j . We consider the class of CFP's for which the ratio of the transition rates has the form

$$(1.4) \quad \frac{\psi(i, j)}{\phi(i, j)} = \frac{a_{i+j}}{a_i a_j}, \quad i, j : 2 \leq i + j \leq N,$$

where $a_k > 0$, $k = 1, \dots, N$ are given parameters of the process. Owing to (1.4), the condition of detailed balance holds and, consequently, the CFP considered is reversible with respect to the invariant measure (1.2).

Letting $N \rightarrow \infty$, we will be concerned with the relationship between two infinite sequences $\{a_n\}_1^\infty$ and $\{c_n\}_0^\infty$, $c_0 = 1$.

It was conjectured in [5]* that the existence of the limit

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} > 0$$

implies the existence of the limit

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}} > 0.$$

Apart from the fact that the conjecture is a challenging mathematical problem, one can see from [5] that it also has a direct significance for the stochastic model in question. First, if the limit (1.6) exists, then a variety of functionals of the process (e.g., the expected values and variances of finite group sizes), as $N \rightarrow \infty$, can be explicitly expressed via this limit. Next, by formula (4.16) in [5] we have that

$$(1.7) \quad \text{cov}(n_k, n_l) = a(k)a(l) \left(\frac{c_{N-k-l}}{c_N} - \frac{c_{N-k}c_{N-l}}{(c_N)^2} \right), \\ k \neq l = 1, 2, \dots, N, \quad k + l \leq N.$$

Thus, the validity of the conjecture will imply that at the steady state the random variables n_k, n_l , $k \neq l$ become uncorrelated, as $N \rightarrow \infty$. This fact incorporates into the assumption of independence of sites in mean-field models, as $N \rightarrow \infty$, that is commonly accepted in statistical physics.

Another motivation for our study is provided by a quite different field, known as random combinatorial structures (RCS's). The connection of CFP's to this field is based on the following observation made in [5]. Let Z_i , $i = 1, \dots, N$ be independent Poisson random variables with respective means $a_i > 0$, $i = 1, \dots, N$. Then it is easy to see that the distribution μ_N admits the following representation:

$$(1.8) \quad \mu_N(\eta) = \Pr\{Z_1 = n_1, \dots, Z_N = n_N \mid \sum_{i=1}^N iZ_i = N\}, \\ \eta = (n_1, \dots, n_N) \in \Omega_N.$$

It turns out that (1.8) is the general form of distributions arising in a variety of RCS's. This is explained in [1], [2] and [11]. (Theorem 1, p. 96 in [1] gives a rigorous proof of this fact.) The simplest example of a RCS is a random

* *Added in proofs:* The conjecture was recently proved in the setting of general additive number systems by J. Ball and S. Burris in *Asymptotics for Logical Limit Laws*, preprint, 2001.

choice from $N!$ permutations of N objects. Cauchy's formula for the number of permutations having n_k cycles of length k , $k = 1, \dots, N$, where $\sum_{k=1}^N kn_k = N$, tells us that the probability of picking a permutation with this property is given by (1.2) with $a_k = k^{-1}$, $k = 1, \dots, N$ and $C_N = 1$.

In view of this, one can translate the preceding reasoning in the context of the Conjecture, into the language of RCS's.

Our paper is devoted exclusively to the study of the asymptotic behaviour, as $n \rightarrow \infty$, of the quantity c_n , defined by (1.3).

The asymptotic formula for c_n established in our paper proves the Conjecture for a wide class of parameter functions a : $a(k) = a_k$, $k = 1, 2, \dots$. We mention also two other applications of our result related to global characteristics of CFP's (=RCS's).

(i) For a given n , denote by v_n the mean value of the total number of different groups at the equilibrium of CFP (=components in a RCS). It follows from (4.15) in [5] that

$$(1.9) \quad v_n = \sum_{k=1}^n a_k \frac{c_{n-k}}{c_n}.$$

(ii) Denote by p_∞ the probability at the steady state of the creation of a cluster (= component in a RCS) of infinite size. It was shown in [5], p. 462, that the condition

$$(1.10) \quad \lim_{n \rightarrow \infty} (v_n - v_{[\alpha n]-1}) > 0$$

for some $0 < \alpha \leq 1$ is sufficient for $p_\infty > 0$. (Here $[\bullet]$ is the integer part of a number.)

Thus, with the help of the asymptotic formula for c_n , one can reveal the asymptotic behaviour of v_n as $n \rightarrow \infty$ and, consequently, find the limit in (1.10). The latter will answer a question which is common in statistical physics.

Also, we want to point out that determining the asymptotic properties of partition functions for interacting particle systems is a difficult mathematical problem widely discussed in statistical physics (see, e.g., [15]).

2. Description of the method and a sketch of its history

We assume $a_n > 0$, $n = 1, 2, \dots$ and that the following limit exists:

$$(2.11) \quad \lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} := R, \quad 0 < R < \infty.$$

Thus the power series in x ,

$$(2.12) \quad S(x) := \sum_{n=1}^{\infty} a_n x^n,$$

has radius of convergence R and it converges in the complex domain $D \subseteq \{|x| \leq R\}$.

Then (see [5]), $g(x) = e^{S(x)}$, $x \in D$ is the generating function for the sequence $\{c_n\}_0^\infty$ defined by (1.3). Namely,

$$(2.13) \quad g(x) = e^{S(x)} = \sum_{n=0}^{\infty} c_n x^n, \quad x \in D,$$

and, moreover, the series (2.12) and (2.13) converge in the same domain D .

The method we use here for deriving the asymptotic formula for c_n goes back to A. Khintchine's pioneering monograph [10]. In [10] Khintchine developed the idea of expressing values of quantum statistics via the probability function of a sum of correspondingly constructed independent integer-valued random variables. Subsequent implementation of the local limit theorem resulted in the method of the derivation of asymptotic distributions of quantum statistics. In [10] this method was systematically applied to systems of photons and some other models. The method was further developed by A. Postnikov and G. Freiman (see [14] for references) who applied it to analytic number theory. In particular, G. Freiman formulated a local limit theorem for some asymptotic problems related to partitions. A general scheme for the derivation of asymptotic formulae for these kind of problems was outlined by G. Freiman and J. Pitman in [8], [7] (see also [4] for references.)

A similar approach, also based on the implementation of the local limit theorem, has been developed independently for the last fifteen years in the theory of RCS's. A very good exposition of this direction of research is given in the recent monograph [11] by V. Kolchin. We will explain briefly the basic difference between the problem addressed in the present paper and those in [11]. In the context of the generalized scheme of allocation that encompasses a variety of RCS's, S and g are the generating functions for, respectively, the total number of combinatorial objects of size n and for the number of such objects possessing a definite property. In this setting it is assumed that the expression for the function g is known explicitly. Based on this, the combinatorial quantity in question is expressed via the probability function of a sum of i.i.d. discrete random variables, distributed according to a probability law that depends on the given

values of $\{c_n\}_0^\infty$. Such a scheme is applicable, for example, to the investigation of the asymptotic of the number $F_{n,N}$ of all forests of N nonrooted trees having n vertices, in which case $c_n = (n!)^{-1}n^{n-2}$, $n = 1, 2, \dots$

First, w.l.o.g. we assume throughout the paper that the common radius of convergence of the series (2.12) and (2.13) equals 1. This is due to the fact that taking in (1.3) $\tilde{a}_j = R^j a_j$, $j = 1, \dots, N$, it follows from (1.1) that $\tilde{c}_N = R^N c_N$. The above assumption makes the $\lim_{n \rightarrow \infty} (a_n/a_{n+1})$, if it exists, equal to 1.

Our starting point is the following representation of c_n .

LEMMA 1:

$$(2.14) \quad c_n = e^{n\sigma} \int_0^1 \prod_{l=1}^n \left(\sum_{k=0}^{\infty} \frac{a_l^k e^{-lk\sigma + 2\pi i \alpha l k}}{k!} \right) \times e^{-2\pi i \alpha n} d\alpha, \quad n = 1, 2, \dots$$

for any real σ .

Proof: It follows from (1.3) that c_n depends only on $a_1, \dots, a_n$, which means that the first $n+1$ terms of the Taylor series expansions of the two functions

$$(2.15) \quad g(x) = e^{S(x)} \quad \text{and} \quad g_n(x) := e^{\sum_{l=1}^n a_l x^l}, \quad x \in D$$

are the same, i.e., $c_k = c_{k,n}$, $k = 0, \dots, n$, where $\{c_{k,n}\}_{k=0}^\infty$ is the sequence related to the function g_n . For a fixed n , the series expansion of the function $g_n(x)$ converges for all x . So, we can set

$$(2.16) \quad x = e^{-\sigma + 2\pi i \alpha},$$

for some real σ and α .

Then we have

$$(2.17) \quad \int_0^1 g_n(x) e^{-2\pi i \alpha n} d\alpha = \int_0^1 \left(\sum_{k=0}^{\infty} c_{k,n} e^{-\sigma k + 2\pi i \alpha (k-n)} \right) d\alpha = c_n e^{-n\sigma}.$$

The last equality is due to the fact that

$$(2.18) \quad \int_0^1 e^{2\pi i \alpha m} d\alpha = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0, \end{cases} \quad m \in \mathbb{Z}.$$

Finally, substituting

$$(2.19) \quad g_n(x) = \prod_{l=1}^n e^{a_l x^l} = \prod_{l=1}^n \left(\sum_{k=0}^{\infty} \frac{(a_l x^l)^k}{k!} \right)$$

and (2.16) in the LHS of (2.17) we get the claim. ■

Our next step will be to give a probabilistic meaning to the expression (2.14) for c_n .

We introduce the following notations:

$$(2.20) \quad S_n(x) = \sum_{l=1}^n a_l x^l,$$

$$(2.21) \quad p_{lk} = \frac{a_l^k e^{-\sigma lk}}{k! \exp(a_l e^{-\sigma l})}, \quad l = 1, \dots, n, \quad k = 0, 1, \dots,$$

$$(2.22) \quad \varphi_l(\alpha) = \sum_{k=0}^{\infty} p_{lk} e^{2\pi i \alpha k}, \quad \alpha \in R,$$

$$(2.23) \quad \varphi(\alpha) = \prod_{l=1}^n \varphi_l(\alpha), \quad \alpha \in R.$$

Now (2.14) can be rewritten as

$$(2.24) \quad c_n = e^{n\sigma} e^{S_n(e^{-\sigma})} \int_0^1 \varphi(\alpha) e^{-2\pi i \alpha n} d\alpha.$$

The fact that for a given l ($1 \leq l \leq n$), p_{lk} , $k = 0, 1, \dots$ is a Poisson probability function with parameter $a_l e^{-\sigma l}$, suggests the following probabilistic interpretation of the integral in the RHS of (2.24).

Let $X_1, \dots, X_n$ be independent integer-valued random variables defined by

$$(2.25) \quad \Pr(X_l = lk) = p_{lk}, \quad l = 1, \dots, n, \quad k = 0, 1, \dots$$

Then $\varphi(\alpha)$ defined above is the characteristic function of the sum $Y = X_1 + \dots + X_n$ and we have

$$(2.26) \quad \int_0^1 \varphi(\alpha) e^{-2\pi i \alpha n} d\alpha = \Pr(Y = n).$$

Now (2.24) can be viewed as an analog of the aforementioned Khintchine representation for c_n .

It is well-known [9] from the classical theory of limit distributions of sums of independent integer-valued random variables that, under certain conditions on distributions of the variables, a local limit theorem is valid.

In our subsequent study, the free parameter σ will be taken depending on n : $\sigma = \sigma_n$. By (2.25) and (2.21) this means that the probability law of each of the n random variables X_l , $l = 1, 2, \dots, n$ depends on n . Therefore, to compare with the classical case, we will be dealing here with a triangular array of random

variables. For this case, general necessary and sufficient conditions for validity of the local limit theorem are not known. For some cases results in this direction were obtained in [13] and [3]. In the first of these two papers a sufficient condition was established (see ([13], Theorem 2, condition III) in the case of an array of general lattice random variables. It can be verified that this condition (which can be viewed as a version of the celebrated condition of asymptotic uniformity) fails for the class of parameter functions a considered in our paper. The second paper studies exclusively the case of the triangular array of trinomial random variables.

Most of this paper is devoted to the proof of the local limit theorem in the above setting. Namely, we will demonstrate that under certain conditions on the parameter function a

$$(2.27) \quad \Pr(Y = n) \sim (2\pi B_n^2)^{-1/2} e^{-(M_n - n)^2 / 2B_n^2}, \quad \text{as } n \rightarrow \infty,$$

where $M_n = EY$ and $B_n^2 = \text{Var } Y$.

3. Proof of the local limit theorem

In order to prove (2.27) we have to find the asymptotic formula, as $n \rightarrow \infty$, for the integral in the RHS of (2.24). We will denote constants in the sequel by $\gamma, \gamma_i, i = 1, 2, \dots$

First, we obtain the explicit expressions for the quantities M_n and B_n^2 . Expressions (2.25) and (2.21) say that $l^{-1}X_l, l = 1, \dots$, are Poisson $(a_l e^{-\sigma l})$ random variables. So, we have

$$(3.28) \quad EX_l = la_l e^{-\sigma l}, \quad l = 1, \dots, n,$$

$$(3.29) \quad \text{Var } X_l = l^2 a_l e^{-l\sigma}, \quad l = 1, \dots, n.$$

This gives

$$(3.30) \quad M_n = \sum_{l=1}^n la_l e^{-l\sigma}, \quad n = 1, 2, \dots,$$

$$(3.31) \quad B_n^2 = \sum_{l=1}^n l^2 a_l e^{-l\sigma}, \quad n = 1, 2, \dots$$

It follows from the preceding discussion that the representation (2.24) holds for any real σ . Our next result shows that σ can be chosen so that the exponential factor in the RHS of (2.27) equals 1, for any $n = 1, 2, \dots$

LEMMA 2: *The equation*

$$(3.32) \quad \sum_{l=1}^n la_l e^{-l\sigma} = n$$

has a unique solution $\sigma = \sigma_n$, for any $n = 1, 2, \dots$

Proof: The assertion follows immediately from the assumption $a_l > 0$, $l = 1, 2, \dots$ ■

Remark: The above choice of the free parameter σ makes the probability of the event $\{Y = n\}$ large, as $n \rightarrow \infty$. The same idea is widely used for approximation of RCS's by independent processes (see [1] for references). In statistical physics, such a way of choosing a free parameter for estimating averages is known as the Darwin–Fowler method developed in the 1930's (see, e.g., [6] for references).

It follows from (3.32) that if the series $\sum_{l=1}^{\infty} la_l$ converges, then $\sigma_n \leq 0$ for sufficiently large n , while in the opposite case the sign of σ_n depends on the behaviour of $S'_n(1)$, as $n \rightarrow \infty$. However, in both cases the following basic property of σ_n , $n = 1, 2, \dots$ holds.

LEMMA 3: *Let*

$$(3.33) \quad \lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = 1.$$

Then $\lim_{n \rightarrow \infty} \sigma_n = 0$.

Proof: By the definition of σ_n ,

$$(3.34) \quad n^{-1} \sum_{l=1}^n la_l e^{-l\sigma_n} = 1, \quad n = 1, 2, \dots$$

Denote $b_l = la_l > 0$, $l = 1, 2, \dots$. Based on (3.33) let $N = N(\epsilon)$, where $\epsilon > 0$ is such that

$$(3.35) \quad 1 - \epsilon \leq \frac{b_{l+1}}{b_l} \leq 1 + \epsilon, \quad \text{for all } l \geq N.$$

Consequently,

$$(3.36) \quad (1 - \epsilon)^{l-N} b_N \leq b_l \leq (1 + \epsilon)^{l-N} b_N, \quad l \geq N.$$

Now suppose that $\lim_{k \rightarrow \infty} \sigma_{n_k} = \sigma$, for some subsequence $n_k \rightarrow +\infty$, as $k \rightarrow \infty$, where $|\sigma| \leq \infty$. Let first $-\infty \leq \sigma < 0$; then taking $0 < \epsilon < 1 - e^{\sigma/2}$ we

have

$$(3.37) \quad n_k^{-1} \sum_{l=1}^{n_k} b_l e^{-l\sigma_{n_k}} \geq (1-\epsilon)^{-N} b_N n_k^{-1} \sum_{l=N}^{n_k} [(1-\epsilon)e^{-\sigma_{n_k}}]^l \rightarrow \infty, \text{ as } k \rightarrow \infty$$

since

$$(1-\epsilon)e^{-\sigma_{n_k}} > e^{\sigma/2 - \sigma_{n_k}} \rightarrow e^{-\sigma/2} > 1, \text{ as } k \rightarrow \infty.$$

If now $0 < \sigma \leq \infty$, then for $0 < \epsilon < e^{\sigma/2} - 1$,

$$(3.38) \quad n_k^{-1} \sum_{l=1}^{n_k} b_l e^{-l\sigma_{n_k}} \leq n_k^{-1} \left(\sum_{l=1}^N b_l e^{-l\sigma_{n_k}} + (1+\epsilon)^{-N} b_N \sum_{l=N}^{n_k} ((1+\epsilon)e^{-\sigma_{n_k}})^l \right) \rightarrow 0, \\ \text{as } k \rightarrow \infty,$$

since in this case $e^{-\sigma/2} < 1$. Both (3.37) and (3.38) contradict (3.34), which implies that $\limsup_{n \rightarrow \infty} \sigma_n = \liminf_{n \rightarrow \infty} \sigma_n = 0$. ■

In what follows we will assume that $\sigma = \sigma_n$, as defined by (3.32). Our next lemma provides the expression for the integrand in the LHS of (2.26) for small values of α .

LEMMA 4: For a fixed n and $\sigma = \sigma_n$,

$$(3.39) \quad \varphi(\alpha) e^{-2\pi i \alpha n} = \exp(-2\pi^2 \alpha^2 B_n^2 + O(\alpha^3 \rho_3)), \text{ as } \alpha \rightarrow 0,$$

where $\rho_3 = \rho_3(n) = \sum_{l=1}^n l^3 a_l e^{-l\sigma_n}$, $n = 1, 2, \dots$

Proof: By (2.21)–(2.23),

$$(3.40) \quad \varphi_l(\alpha) = \sum_{k=0}^{\infty} \frac{(a_l e^{-\sigma_n l} e^{2\pi i \alpha l})^k}{k! \exp(a_l e^{-l\sigma_n})} = \exp(a_l e^{-l\sigma_n} (e^{2\pi i \alpha l} - 1)), \quad \alpha \in R$$

and

$$(3.41) \quad \varphi(\alpha) = \exp\left(\sum_{l=1}^n a_l e^{-l\sigma_n} (e^{2\pi i \alpha l} - 1)\right), \quad \alpha \in R.$$

Finally, substituting in (3.41) the Taylor expansion (in α)

$$(3.42) \quad e^{2\pi i \alpha l} - 1 = 2\pi i \alpha l - 2\pi^2 \alpha^2 l^2 + O(\alpha^3 l^3), \text{ as } \alpha \rightarrow 0,$$

which holds uniformly for $l \geq 1$, and making use of the definition (3.32) of σ_n , proves (3.39).

Observe that uniformity (=the constant implied by the term $O(\alpha^3 l^3)$ in (3.42) does not depend on l) is due to the fact that for any real α ,

$$\left| \frac{d^s}{d\alpha^s} e^{2\pi i \alpha l} \right| \leq (2\pi)^s l^s, \quad s = 1, 2, \dots \quad \blacksquare$$

Now we are prepared to deal with the central objective stated in the beginning of this section. Denote

$$(3.43) \quad T = T(n) = \int_0^1 \varphi(\alpha) e^{-2\pi i \alpha n} d\alpha, \quad n = 1, 2, \dots$$

The integrand in (3.43) is periodic with period 1. So for any $0 < \alpha_0 \leq 1/2$, the integral T can be written as

$$(3.44) \quad T = T_1 + T_2,$$

where $T_1 = T_1(\alpha_0; n)$ and $T_2 = T_2(\alpha_0; n)$ are integrals of the integrand in (3.43) over the sets $[-\alpha_0, +\alpha_0]$ and $[-1/2, -\alpha_0] \cup [\alpha_0, 1/2]$, respectively. Following the idea of [7], [8], we will first show that for an appropriate choice of $\alpha_0 = \alpha_0(n)$ the main contribution to T , as $n \rightarrow \infty$, comes from T_1 . Then, estimating T_1 , under $\alpha_0 = \alpha_0(n)$, $n \rightarrow \infty$ we will get the desired asymptotic formula (2.27).

It is clear from Lemma 4 that the asymptotic behaviour of the integral T_1 as $n \rightarrow \infty$ is determined by the asymptotics of the three key parameters σ_n , B_n^2 and ρ_3 .

First, we address the problem for the class of parameter functions a of the form

$$(3.45) \quad a_j = j^{p-1}, \quad j = 1, 2, \dots \quad \text{for a given } p > 0.$$

It follows from the definition of the above three parameters that in the case considered the problem reduces to the estimation of sums of the form $\sum_{j=1}^n j^k e^{-\sigma j}$, $k > -1$, as $n \rightarrow \infty$.

To do this we apply the integral test for the function

$$(3.46) \quad f(u) = (\sigma u)^k e^{-\sigma u}, \quad u \geq 0, \quad \sigma \geq 0, \quad k > -1.$$

In the case $k > 0$ the function f is strictly increasing on $[0, k\sigma^{-1}]$ and is strictly decreasing on $[k\sigma^{-1}, +\infty)$. So, applying the integral test separately on each of the above intervals we have in the case considered

$$(3.47) \quad \int_0^n f(u) du + \gamma_1 f(k\sigma^{-1}) \leq \sum_{j=1}^n f(j) \leq \int_1^n f(u) du + \gamma_2 f(k\sigma^{-1}),$$

where the constants γ_1, γ_2 depend on k only and $f(k\sigma^{-1}) = k^k e^{-k}$.

If now $-1 < k \leq 0$, then the function f is strictly decreasing on $[0; +\infty)$ and the integral test gives

$$(3.48) \quad \int_1^{n+1} f(u) du \leq \sum_{j=1}^n f(j) \leq \int_0^n f(u) du.$$

Note also that for any $\sigma, b > 0$

$$(3.49) \quad \int_b^n f(u) du = \sigma^k \int_b^n u^k e^{-\sigma u} du = \sigma^{-1} \int_{b\sigma}^{n\sigma} z^k e^{-z} dz.$$

Then, combining (3.46)–(3.49) we arrive at the desired asymptotic estimate:

$$(3.50) \quad \sum_{j=1}^n j^k e^{-\sigma j} \sim \frac{\gamma}{\sigma^{k+1}}, \quad \text{as } 0 < \sigma \rightarrow 0, \quad n \rightarrow \infty, \quad \text{and} \quad \liminf_{n \rightarrow \infty} n\sigma > 0.$$

In particular, if $n\sigma \rightarrow \infty$, then the constant γ in (3.50) can be found explicitly:

$$(3.51) \quad \sum_{j=1}^n j^k e^{-\sigma j} \sim \frac{\Gamma(k+1)}{\sigma^{k+1}},$$

where Γ is the gamma function.

Further, we will write $\bullet(n) \asymp n^\alpha$ if there exist positive constants γ_1, γ_2 , s.t. $\gamma_1 n^\alpha \leq \bullet(n) \leq \gamma_2 n^\alpha$, for all sufficiently large n .

Extending (3.45) we consider now the class of functions a satisfying

$$(3.52) \quad a_j \asymp j^{p-1}, \quad \text{as } j \rightarrow \infty \quad \text{for a given } p > 0.$$

An obvious variation of the preceding argument gives in this case the following analog of (3.50):

$$(3.53) \quad \sum_{j=1}^n a_j j^l e^{-\sigma j} \asymp \frac{1}{\sigma^{p+l}}, \quad \text{as } 0 < \sigma \rightarrow 0, \quad n \rightarrow \infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} n\sigma > 0, \quad l \geq 0.$$

This immediately implies

LEMMA 5: *Let the function a obey (3.52). Then, as $n \rightarrow \infty$,*

$$(3.54) \quad \sigma_n \asymp n^{-1/(p+1)}$$

$$(3.55) \quad B_n^2 \asymp n^{(p+2)/(p+1)}$$

and

$$(3.56) \quad \rho_3 \asymp n^{(p+3)/(p+1)}.$$

Proof: By the definition of σ_n , Lemma 3 and (3.53), in this case

$$(3.57) \quad n \asymp \frac{1}{\sigma_n^{p+1}}, \quad \text{as } n \rightarrow \infty$$

and, consequently,

$$(3.58) \quad \sigma_n \asymp n^{-1/(p+1)}, \quad \text{as } n \rightarrow \infty.$$

Now the last two assertions follow from (3.53). $\blacksquare$

At this point we are prepared to estimate the integral

$$(3.59) \quad T_1 = T_1(\alpha_0; n) = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e^{-2\pi i \alpha n} d\alpha.$$

LEMMA 6: *Let the function a obey (3.52) and $\alpha_0 = \sigma_n^{(p+2)/2} \log n$, $p > 0$. Then*

$$(3.60) \quad T_1(\alpha_0; n) \sim (2\pi B_n^2)^{-1/2}, \quad \text{as } n \rightarrow \infty.$$

Proof: By (3.54), $\alpha_0 \rightarrow 0$, as $n \rightarrow \infty$. So, making use of (3.39) we obtain

$$(3.61) \quad T_1 = \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 B_n^2 + O(\alpha^3 \rho_3)) d\alpha, \quad \text{as } n \rightarrow \infty.$$

Further, under the condition (3.52), it follows from Lemma 5 that the above choice of α_0 provides the following two basic relationships between the parameters B_n^2 and ρ_3 :

$$(3.62) \quad \lim_{n \rightarrow \infty} \alpha_0^2 B_n^2 = \lim_{n \rightarrow \infty} \log^2 n = +\infty$$

and

$$(3.63) \quad \lim_{n \rightarrow \infty} \alpha_0^3 \rho_3 = \lim_{n \rightarrow \infty} \sigma_n^{p/2} \log^3 n = 0.$$

Since $\lim_{n \rightarrow \infty} \alpha^3 \rho_3 = 0$ for all $\alpha \in [-\alpha_0, \alpha_0]$, (3.61) and (3.62) imply

$$(3.64) \quad \begin{aligned} T_1 &\sim \int_{-\alpha_0}^{\alpha_0} \exp(-2\pi^2 \alpha^2 B_n^2) d\alpha = \frac{1}{2\pi B_n} \int_{-2\pi \alpha_0 B_n}^{2\pi \alpha_0 B_n} \exp(-z^2/2) dz \\ &\sim 1/\sqrt{2\pi B_n^2}, \quad \text{as } n \rightarrow \infty. \quad \blacksquare \end{aligned}$$

Taking α_0 as in Lemma 6, we write

$$(3.65) \quad T_2 = T_{2,1} + T_{2,2},$$

where

$$(3.66) \quad T_{2,1} = T_{2,1}(\alpha_0; n) := \int_{\alpha_0}^{1/2} \varphi(\alpha) e^{-2\pi i \alpha n} d\alpha$$

and $T_{2,2}$ is the integral of the same integrand, but over the set $[-1/2, \alpha_0]$.

In view of (3.65) and the fact that $\varphi(-\alpha) = \varphi(\alpha)$, $\alpha \in R$, the rest of this section is devoted to an estimation of the integral $T_{2,1}$, as $n \rightarrow \infty$. Our starting argument will be the same as in [8]. It follows from (3.41) that

$$(3.67) \quad |\varphi(\alpha)| = \exp\left(-2 \sum_{j=1}^n a_j e^{-j\sigma_n} \sin^2 \pi \alpha j\right), \quad \alpha \in R.$$

Denote by $[x]$ and $\{x\}$ respectively the integer and fractional parts of a real number x , and by $\|x\|$ the distance from x to the nearest integer, so that

$$(3.68) \quad \|x\| = \begin{cases} \{x\}, & \text{if } \{x\} \leq 1/2, \\ 1 - \{x\}, & \text{if } \{x\} > 1/2. \end{cases}$$

We will make use of the inequality

$$(3.69) \quad \sin \theta \geq \frac{2}{\pi} \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Since $\sin^2 \pi x = \sin^2 \pi \|x\|$ for any real x , it follows from (3.69) and (3.68) that for all real x ,

$$(3.70) \quad \sin^2 \pi x \geq 4\|x\|^2.$$

Hence, in view of (3.67) and (3.66) we have to estimate the sum

$$(3.71) \quad V_n(\alpha) := \sum_{j=1}^n a_j e^{-j\sigma_n} \|\alpha j\|^2, \quad \alpha_0 \leq \alpha \leq 1/2.$$

LEMMA 7: *Let the function a obey (3.52). Then*

$$(3.72) \quad V_n(\alpha) \geq \gamma \log^2 n, \quad \alpha \in [\alpha_0, 1/2], \quad \text{as } n \rightarrow \infty,$$

where $\gamma > 0$.

Proof: Take $\alpha_1 = \alpha_1(n) = \sigma_n > \alpha_0(n)$, as $n \rightarrow \infty$, and split the interval $[\alpha_0; 1/2]$ into two disjoint subintervals $I_1 := [\alpha_0, \alpha_1]$ and $I_2 := (\alpha_1, 1/2]$. We plan to prove the assertion separately for $\alpha \in I_1$ and $\alpha \in I_2$.

INTERVAL I_1 . It is clear from the definition (3.68) that

$$(3.73) \quad \|\alpha j\| = \alpha j, \quad j \leq 1/2\alpha_1, \quad \alpha \in I_1.$$

In view of the fact that, by Lemma 3, $(2\alpha_1)^{-1} \rightarrow \infty$, as $n \rightarrow \infty$, while $(\alpha_1)^{-1}\sigma_n = 1$, $n = 1, 2, \dots$, we apply (3.53) with $l = 2$ to obtain

$$(3.74) \quad V_n(\alpha) \geq \alpha_0^2 \sum_{1 \leq j \leq (2\alpha_1)^{-1}} a_j j^2 e^{-j\sigma_n} \asymp \alpha_0^2 / \sigma_n^{p+2} = \log^2 n, \quad \alpha \in I_1.$$

INTERVAL I_2 . For a given integer n and a given $\alpha \in I_2$ define the set of integers

$$Q(\alpha) = Q(\alpha; n) = \{1 \leq j \leq n: \|\alpha j\| \geq 1/4\}.$$

It is clear that

$$(3.75) \quad \begin{aligned} Q(\alpha) &= \{1 \leq j \leq n: k + 1/4 \leq \alpha j \leq k + 3/4, \quad k = 0, 1, \dots\} \\ &= \bigcup_{k=0}^{[(4\alpha n - 3)/4]} Q_k(\alpha), \end{aligned}$$

where $Q_k(\alpha)$ denotes the set of integers $\{j: (4k+1)/4\alpha \leq j \leq (4k+3)/4\alpha\}$.

Observe that for any $\alpha \in I_2$ and $k \geq 0$ the set $Q_k(\alpha)$ is not empty, since in this case $(4k+3)/4\alpha - (4k+1)/4\alpha \geq 1$.

This yields the following estimate of the sum $V_n(\alpha)$, $\alpha \in I_2$:

$$(3.76) \quad V_n(\alpha) \geq 1/16 \sum_{j \in Q(\alpha)} a_j e^{-j\sigma_n} = 1/16 \sum_{k=0}^{[(4\alpha n - 3)/4]} \sum_{j \in Q_k(\alpha)} a_j e^{-j\sigma_n}.$$

We now assume that the asymptotic inequality (3.52) holds for all $j \geq N$. This means that (3.52) is valid for all $j \in Q_k(\alpha)$ whenever $k \geq \max\{0; (4\alpha N - 1)/4\} =: K(\alpha)$.

We agree, with an obvious abuse of notation, that for a real u , $Q_u(\alpha)$ is the interval

$$\left[\frac{4u+1}{4\alpha}, \frac{4u+3}{4\alpha} \right].$$

Observe that for all sufficiently large n , we have $0 < \alpha^{-1}\sigma_n \leq 1$, $\alpha \in I_2$, while $n\sigma_n \rightarrow \infty$.

Applying now the integral test to the double sum in the RHS of (3.76) gives

$$V_n(\alpha) \geq \gamma \int_{K(\alpha)}^{(4\alpha n - 3)/4} du \int_{v \in Q_u(\alpha)} v^{p-1} e^{-v\sigma_n} dv$$

$$\begin{aligned}
&= \gamma \frac{1}{\sigma_n^p} \int_{K(\alpha)}^{(4\alpha n-3)/4} du \int_{v \in (\sigma_n Q_u(\alpha))} v^{p-1} e^{-v} dv \\
&\geq \gamma \frac{1}{\sigma_n^p} \int_{K_1(\alpha)}^{n\sigma_n - \sigma_n/2\alpha} v^{p-1} e^{-v} dv \int_{\alpha v/\sigma_n - 3/4}^{\alpha v/\sigma_n - 1/4} du \\
(3.77) \quad &\asymp \frac{1}{\sigma_n^p}, \quad p > 0, \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

where we denote

$$K_1(\alpha) = \frac{4K(\alpha) + 3}{4\alpha} \sigma_n \quad \text{and} \quad \sigma_n(Q_u(\alpha)) = \left[\sigma_n \frac{4u+1}{4\alpha}, \sigma_n \frac{4u+3}{4\alpha} \right].$$

Note that the last inequality in (3.77) is obtained via the change of the order of integration. Finally, (3.77), (3.54) and (3.74) prove the claim. ■

The last statement of this section is the desired local limit theorem.

THEOREM 1: *Let the function a obey (3.52). Then*

$$(3.78) \quad T = \Pr(Y = n) \sim 1/\sqrt{2\pi B_n^2}, \quad \text{as } n \rightarrow \infty.$$

Proof: By Lemma 7 and (3.66), (3.67) and (3.65), we have

$$(3.79) \quad T_2 \leq e^{-2\gamma \log^2 n}, \quad \text{as } n \rightarrow \infty,$$

where $\gamma > 0$. In view of (3.64) and (3.44) this proves (3.78). ■

We provide now an extension of the field of validity of the above local limit theorem. We agree to write $n^{\beta_1} \preceq \bullet(n) \preceq n^{\beta_2}$, $0 \leq \beta_1 \leq \beta_2$, if there exist positive constants γ_1, γ_2 , s.t. $\gamma_1 n^{\beta_1} \leq \bullet(n) \leq \gamma_2 n^{\beta_2}$, for all sufficiently large n .

For given $0 < p_1 \leq p_2$ define the set $\mathcal{F}(p_1, p_2)$ of parameter functions $a = a(j)$, $j \in R^+$, obeying (3.33) and the condition

$$(3.80) \quad j^{p_1-1} \preceq a(j) \preceq j^{p_2-1}, \quad 0 < p_1 \leq p_2.$$

COROLLARY 1: *For an arbitrary $p > 0$ and $0 < \epsilon \leq p/3$ the local limit theorem (3.78) is valid for all parameter functions $a \in \mathcal{F}(2p/3 + \epsilon; p)$.*

Proof: It is clear from the preceding results that for all $a \in \mathcal{F}(p_1; p_2)$, we must have, as $n \rightarrow \infty$,

$$\begin{aligned}
(3.81) \quad &n^{-1/(p_1+1)} \preceq \sigma_n \preceq n^{-1/(p_2+1)} \\
&\sigma_n^{-(p_1+2)} \preceq B_n^2 \preceq \sigma_n^{-(p_2+2)}, \quad \sigma_n^{-(p_1+3)} \preceq \rho_3 \preceq \sigma_n^{-(p_2+3)}.
\end{aligned}$$

Therefore setting, as in Lemma 6, $\alpha_0 = (B_n)^{-1} \log n$, gives

$$(3.82) \quad \alpha_0^3 \rho_3 \leq \gamma_3 (\log^3 n) \sigma_n^{3(p_1+2)/2} \sigma_n^{-(p_2+3)} = \gamma_3 (\log^3 n) \sigma_n^{(3p_1-2p_2)/2} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

provided $3p_1 - 2p_2 > 0$.

Thus, in the case $a \in \mathcal{F}(p_1; p_2)$, where $p_1, p_2 : 3p_1/2 > p_2 \geq p_1 > 0$, Lemma 6 is valid. The proof of Lemma 7 for this case goes along the same lines, with an obvious replacement of (3.77) by

$$(3.83) \quad 1/\sigma_n^{p_1} \preceq V_n(\alpha) \preceq 1/\sigma_n^{p_2}, \quad \alpha \in I_2.$$

Combining these results proves the validity of the local limit theorem for the class of functions a in our statement. ■

For our subsequent study we will need the following extension of (3.78).

COROLLARY 2: *Under the conditions of Corollary 1,*

$$(3.84) \quad \Pr(Y = n + h) \sim \Pr(Y = n) \sim 1/\sqrt{2\pi B_n^2}, \quad \text{as } n \rightarrow \infty,$$

for a fixed real h .

Proof: By (2.26),

$$(3.85) \quad \bar{T}(h; n) := \Pr(Y = n + h) = \int_{-1/2}^{1/2} \varphi(\alpha) e^{-2\pi i \alpha(n+h)} d\alpha,$$

where the characteristic function $\varphi(\alpha)$ is given by (2.22) and (2.23). By Lemma 4, we get

$$(3.86) \quad \varphi(\alpha) e^{-2\pi i \alpha(n+h)} = \exp(-2\pi^2 \alpha^2 B_n^2 - 2\pi i \alpha h + O(\alpha^3 \rho_3)), \quad \text{as } \alpha \rightarrow 0,$$

where ρ_3 is defined as in (3.39). Next, let $\alpha_0 = \alpha_0(n)$ be as in Lemma 6. Denote

$$(3.87) \quad \bar{T}_1(h; n) = \int_{-\alpha_0}^{\alpha_0} \varphi(\alpha) e^{-2\pi i \alpha(n+h)} d\alpha.$$

Lemma 6 and (3.86) imply for a fixed $h \in R$

$$(3.88) \quad \bar{T}_1(h; n) \sim 1/\sqrt{2\pi B_n^2}, \quad \text{as } n \rightarrow \infty.$$

Now it is left to estimate the integral

$$(3.89) \quad \bar{T}_2(h; n) := \int_{\alpha_0}^{1/2} \varphi(\alpha) e^{-2\pi i \alpha(n+h)} d\alpha + \int_{-1/2}^{-\alpha_0} \varphi(\alpha) e^{-2\pi i \alpha(n+h)} d\alpha.$$

Since the function $\varphi(\alpha)$ here is the same as in (3.67), the estimate (3.79) is valid also for $\bar{T}_2(h; n)$, which together with (3.88) proves the statement. ■

4. The asymptotic formula for c_n

By virtue of (2.24) and Corollary 1, we obtain the following asymptotic formula for c_n valid for all parameter functions $a \in \mathcal{F}(2p/3 + \epsilon; p)$, $p > 0$, $\epsilon > 0$:

$$(4.90) \quad c_n \sim \frac{1}{\sqrt{2\pi B_n^2}} \exp\left(n\sigma_n + \sum_{j=1}^n a_j e^{-j\sigma_n}\right), \quad \text{as } n \rightarrow \infty.$$

Example: Let $a_j \asymp j^{p-1}$, $p > 0$, as $j \rightarrow \infty$. Then by Lemma 5, (3.53) and (4.90),

$$(4.91) \quad \log c_n \asymp 2n^{p/(p+1)} - \frac{1}{2} \log 2\pi - \frac{p+2}{2p+2} \log n, \quad \text{as } n \rightarrow \infty.$$

In particular, if $a_j = j^{p-1}$, $p > 0$, $j = 1, 2, \dots$, then using (3.51), the constants in (3.54) and (3.55) can be found explicitly and we obtain, as $n \rightarrow \infty$,

$$(4.92) \quad \sigma_n \sim \left(\frac{n}{\Gamma(p+1)}\right)^{-1/(p+1)},$$

$$(4.93) \quad B_n^2 \sim \left(\frac{n}{\Gamma(p+1)}\right)^{(p+2)/(p+1)} \Gamma(p+2),$$

$$(4.94) \quad n\sigma_n \sim (\Gamma(p+1))^{1/(p+1)},$$

$$(4.95) \quad \sum_{j=1}^n a_j e^{-j\sigma_n} \sim \Gamma(p) \left(\frac{n}{\Gamma(p+1)}\right)^{1/(p+1)} = p^{-1} (\Gamma(p+1))^{1/(p+1)} n^{p/(p+1)}.$$

Hence, (4.90) gives, as $n \rightarrow \infty$,

$$(4.96) \quad \log c_n \sim A(p) n^{p/(p+1)} - \frac{1}{2} \log 2\pi - \frac{p+2}{2p+2} (\log n - \log \Gamma(p+1)) \\ - \frac{1}{2} \log \Gamma(p+2),$$

where $A(p) = (1 + p^{-1})(\Gamma(p+1))^{1/(p+1)}$.

Remark: For the two cases $a_j = \text{const}$ and $a_j = j$, $j = 1, 2, \dots$ the first (= the principal) term in the asymptotic formula (4.96) was obtained in [5] by solving for large n the corresponding difference equations (4.101) below. Note that this approach is not applicable even for the class of parameter functions $a_j = j^{p-1}$, $p > 0$.

With the help of (4.90) we are able to address the question on the validity of the Conjecture stated in Section 1 (see (1.5) and (1.6)).

ASSERTION: The conjecture is valid for all parameter functions $a \in \mathcal{F}(\frac{2p}{3} + \epsilon; p)$, $p > 0$, $\epsilon > 0$.

Proof: It is clear from (1.3) that the expression for c_{n+1} can be written in the following way:

$$(4.97) \quad c_{n+1} = a_{n+1} + \bar{c}_{n+1},$$

where $\bar{c}_{n+1}$ can be viewed as the value of c_{n+1} (given by (1.3)) when $a_{n+1} = 0$. Next, recall that the representation (2.24) is true for any real σ and all $n = 1, 2, \dots$. We now apply this representation for the particular case $a_{n+1} = 0$. By (2.21) and (2.25), in this case $\Pr(X_{n+1} = 0) = 1$. So, combining (2.24) and (2.26) and taking $\sigma = \sigma_n$ gives

$$(4.98) \quad \bar{c}_{n+1} = \exp\left((n+1)\sigma_n + \sum_{j=1}^n a_j e^{-j\sigma_n}\right) \Pr(Y = n+1),$$

where the random variable Y is defined as in Section 2.

Substituting expression (4.98) in (4.97) and applying (4.90) and Corollary 2, we obtain

$$(4.99) \quad \frac{c_{n+1}}{c_n} \sim e^{\sigma_n} + \frac{a_{n+1}}{c_n}, \quad \text{as } n \rightarrow \infty.$$

In view of the assumption (3.33) and Lemma 3, to complete the proof we have to show that

$$(4.100) \quad \lim_{n \rightarrow \infty} \frac{a_n}{c_n} = 0.$$

To get this result we will implement the difference equation (see [5], p. 460 for references) derived from (2.13):

$$(4.101) \quad \begin{aligned} c_0 &= 1, \quad c_1 = a_1, \\ (n+1)c_{n+1} &= \sum_{j=0}^n (j+1)a_{j+1}c_{n-j}, \quad n = 1, 2, \dots \end{aligned}$$

This gives, for a fixed $k \geq 1$,

$$(4.102) \quad \frac{c_{n+1}}{a_{n+1}} \geq \sum_{j=n-k}^n \frac{(j+1)a_{j+1}c_{n-j}}{(n+1)a_{n+1}}.$$

Consequently, by (3.33) and (4.97)

$$(4.103) \quad \liminf_{n \rightarrow \infty} \frac{c_{n+1}}{a_{n+1}} \geq c_0 + c_1 + \dots + c_k \geq 1 + a_1 + \dots + a_k,$$

for any fixed k .

The desired conclusion now follows from the fact that if $a \in \mathcal{F}(2p/3 + \epsilon; p)$ $p > 0$, $\epsilon > 0$, then

$$(4.104) \quad \sum_{j=1}^{\infty} a_j = \infty. \quad \blacksquare$$

CONCLUDING REMARK. Our asymptotic formula (4.90) is restricted to the case $a \in \mathcal{F}(2p/3 + \epsilon; p)$, $p > 0$, $\epsilon > 0$. The reason we require $p > 0$ comes from the fact that the local limit theorem we proved assumes convergence to the normal law only. This type of convergence is guaranteed if (3.62) and (3.63) hold, or, equivalently, if the parameters ρ_3 and B_n obey the condition

$$(4.105) \quad \lim_{n \rightarrow \infty} \frac{\rho_3}{B_n^3} = 0.$$

Based on the reasoning preceding (3.28), it is not difficult to show that the quantity ρ_3 has the following meaning:

$$(4.106) \quad \rho_3 = \sum_{l=1}^n E(X_l - EX_l)^3.$$

In effect, denoting by U_l the Poisson ($\lambda_l := a_l e^{-\sigma_l}$) random variable, we have

$$(4.107) \quad E(X_l - EX_l)^3 = l^3 E(U_l - EU_l)^3 = l^3 \lambda_l,$$

where the last step is due to the known property of the third central moment of the Poisson distribution (see, e.g., [12], p. 33).

Expression (4.106) explains that (4.105) is Lyapunov's sufficient condition for convergence to the normal law in the central limit theorem. To demonstrate that for $p \leq 0$ the condition (4.105) fails, we consider the case $a_j = j^{-1}$, $j = 1, 2, \dots$. It is easy to see from (3.32) that for this case $\sigma_n = 0$, $n = 1, 2, \dots$. Consequently, we have, as $n \rightarrow \infty$, $B_n^2 \sim n^2$ and $\rho_3 \sim n^3$, which gives $\lim_{n \rightarrow \infty} (\rho_3/B_n^3) = 1$.

The above discussion suggests that for $p \leq 0$ one can expect convergence to other stable laws. The study of this case will require a quite different estimation technique.

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